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SPIRALLING SETS NEAR A HETEROCLINIC NETWORK

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Abstract. There are few explicit examples in the literature of vector fields exhibiting complex dynamics that may be proved analytically. We construct explicitly a vector field on the three-dimensional sphere $\mathbf{S}^3$, whose flow has a spiralling attractor containing the following: two hyperbolic equilibria, heteroclinic trajectories connecting them transversely and a non-trivial hyperbolic, invariant and transitive set. The spiralling set unfolds a heteroclinic network between two symmetric saddle-foci and contains a sequence of topological horseshoes semiconjugate to full shifts over an alphabet with more and more symbols. The vector field is the restriction to $\mathbf{S}^3$ of a polynomial vector field in $\mathbf{R}^4$. In this article, we also identify global bifurcations that induce chaotic dynamics of different types.

Keywords:
Heteroclinic network, Spiralling set, Polynomial vector field

AMS Subject Classifications:
Primary: 34C28; Secondary: 34C37, 37C29, 37D05, 37G35

1. Introduction

Any C^1 vector field defined on a compact three-dimensional manifold may be approximated by a system of differential equations whose flow exhibits one of the following phenomena: uniform hyperbolicity, a heteroclinic cycle associated to, at least, one equilibrium and/or a homoclinic tangency of the invariant manifolds of the periodic solutions — see Arroyo *et al* [9]. However, there are no examples of vector fields exhibiting all three features simultaneously.

In this article, we construct an explicit example of a C^∞ vector field on the three dimensional sphere $\mathbf{S}^3$ that is approximated by differential equations exhibiting all the three behaviours. Nearby differential equations may also display heteroclinic tangencies of invariant manifolds of two equilibria. We describe and characterise some properties of the flow of these differential equations, whose complex geometry arises from spiralling behaviour induced by the presence of saddle-foci. We start with a brief discussion of the literature on examples of this kind.

1.1. **Lorenz-like attractors.** Few explicit examples of vector fields are known, whose flow contains non-hyperbolic invariant sets that are transitive and for which transitivity is robust to small C^1 perturbations. The most famous example is the expanding *butterfly* proposed by E. Lorenz (1963) [19] that arises in a flow having an equilibrium at the origin, where the linearisation has eigenvalues $\lambda_u, \lambda_s^1, \lambda_s^2 \in \mathbf{R}\backslash\{0\}$ satisfying:

$$\lambda_s^2 < \lambda_s^1 < 0 < \lambda_u \quad \text{and} \quad \lambda_u + \lambda_s^1 > 0. \tag{1.1}$$

In a three dimensional manifold, an equilibrium whose linearisation has real eigenvalues satisfying condition (1.1) is what we call an *equilibrium of Lorenz-type*.

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In order to understand the Lorenz differential equations and the phenomenon of robust co-existence in the same transitive set of an equilibrium and regular trajectories accumulating on it, geometric models have been constructed independently by Afraimovich *et al* [1] and Guckenheimer and Williams [15]. The construction of these models have been based on properties suggested by numerics.

Morales *et al* [24] unified the theory of uniformly hyperbolic dynamics and Lorenz-like flows stressing that the relevant notion for the general theory of robustly transitive sets is the *dominated splitting*. More precisely, they have proved that if Λ is a robustly transitive attractor containing at least one equilibrium of Lorenz-type, then Λ must be partially hyperbolic with volume expanding directions, up to reversion of time.

In order to discuss this more rigorously, recall that a compact flow-invariant set Λ is *partially hyperbolic* if there is an invariant splitting $T\Lambda = E^s \oplus E^{cu}$ for which there are $K, \lambda \in \mathbf{R}^+$ such that for $\forall t > 0, \forall x \in \Lambda$:

- $||\partial_x \phi(t,x)|_{E^s_x}|| \leq Ke^{-\lambda t}$;
- $||\partial_x \phi(t,x)|_{E^s_x}||.||\partial_x \phi(t,x)|_{E^{cu}_{\phi(t,x)}}|| \leq Ke^{-\lambda t}$,

where $\phi(t,p)$ is the unique solution $x(t)$ of the initial value problem $\dot{x} = f(x)$, $x(0) = p$, and $f : \mathbf{R}^4 \to \mathbf{R}^4$ is a smooth vector field. The direction E^{cu} of Λ is *volume expanding* if

$$\forall t > 0, \quad \forall x \in \Lambda, \quad det|\partial_x \phi(t,x)|_{E^{cu}_x}| \geq Ke^{\lambda t}.$$

The Lorenz model satisfies these conditions. A good explanation about this subject may be found in chapter 3 of Araújo and Pacífico [6], where it is shown that in a three dimensional manifold the only equilibria that exist near a robustly transitive set must be of Lorenz-type ie, where condition (1.1) holds – see also Bautista [11].

1.2. **Contracting Lorenz Models.** In 1981, Arneodo, Coullet and Tresser [7] started the study of *contracting Lorenz models*. The authors considered a variation of the classical Lorenz model with respect to the eigenvalues at the origin, in which the condition (1.1) is replaced by:

$$\lambda_s^2 < \lambda_s^1 < 0 < \lambda_u \quad \text{and} \quad \lambda_u + \lambda_s^1 < 0. \tag{1.2}$$

In 1993, Rovella [29] proved that there exists a *contracting Lorenz model* $\dot{x} = f(x)$, $x \in \mathbf{R}^3$ with an attractor Λ containing an equilibrium so that the following hold: there exists a local basin of attraction $\mathcal{B}$ of Λ, a neighbourhood U of f (in the C^3– topology) and an open and dense subset $U_0 \subset U$ so that for $\dot{x} = g(x)$ $(g \in U_0)$, the maximal invariant set in $\mathcal{B}$ consists of the equilibria, one or two periodic trajectories, a hyperbolic suspended horseshoe and heteroclinic connections. Moreover, Rovella [29] proved that in generic two parameter families $\dot{x} = f(x, \mu)$, with $f(\star, \overline{0}) \equiv f$, there is a set of positive measure containing $\mu = (0,0)$ for which an attractor in $\mathcal{B}$ containing the equilibrium exists.

The construction of the Rovella attractor [29] is similar to the geometric Lorenz model. Some authors constructed contracting Lorenz-like examples through bifurcations from heteroclinic cycles and networks: for instance, Afraimovich *et al* [2] describe a codimension 1 bifurcation leading from Morse-Smale flows to Lorenz-like attractors; Morales [23] constructed a singular attractor from a hyperbolic flow, through a saddle-node bifurcation. All of these are similar to the expanding Lorenz attractor, for which condition (1.1) hold, as opposed to the contracting case.

1.3. **Spiralling attractors.** Another type of persistent attractor with a much more complex geometry occurs near homoclinic and heteroclinic cycles associated to either a saddle-focus or a non-trivial periodic solution. This complexity is the reason why the study of this subject was left almost untouched for 20 years from L. P. Shilnikov [31, 32] to P. Holmes [16]. Due to the existence of complex eigenvalues of the linearisations at the equilibria, the spiral structure of the non-wandering set predicted by Arneodo, Coullet and Tresser [8] has been observed in some

simulations in the context of the modified Chua's circuit – these attractors are what Aziz-Alaoui [10] call *spiralling attractors.*

Spiralling attractors were expected for perturbations of homoclinic cycles to a saddle-focus under a dissipative condition. There is some evidence that in this case spiral attractors exist and that they might be persistent in the sense of measure [8]. A sequence of topological horseshoes semiconjugate to full shifts on an alphabet with more and more symbols might occur near this kind of attractors.

Nowadays, a particular attention is being given to the study of the dynamics near heteroclinic networks with complex behaviour and multispiral attractors. This creates an interest in the construction of explicit vector fields whose flows have a specific type of invariant set and for which it is possible to give an analytical proof of the properties that guarantee the existence of complex behaviour.

In this article, we describe an explicit example of a polynomial vector field on the three dimensional sphere $\mathbf{S}^3$ whose flow has a heteroclinic network with two saddle-foci, and a spiralling structure containing a hyperbolic non-trivial transitive isolated set. We show that the spiralling set is not robustly transitive, but it presents some similarities to those considered by Rovella [29].

1.4. **Heteroclinic terminology.** Throughout this paper, by *heteroclinic cycle* we mean a set of finitely many disjoint hyperbolic equilibria p_j (also called *nodes*), $j \in \{1, \ldots, k\}$ and trajectories γ_j, $j \in \{1, \ldots, m\}$ such that:

$$\lim_{t \to +\infty} \gamma_j(t) = p_{j+1} = \lim_{t \to -\infty} \gamma_{j+1}(t),$$

called *heteroclinic trajectories*, with the understanding that $\gamma_{m+1} = \gamma_1$ and $p_{k+1} = p_1$. We also allow connected n–dimensional manifolds of solutions biasymptotic to the nodes p_i and p_j, in negative and positive time, respectively. In both cases we denote the connection by $[p_i \to p_j]$. Throughout this article, the nodes are hyperbolic; the dimension of the local unstable manifold of an equilibrium p will be called *the Morse index* of p.

A *heteroclinic network* is a connected set that is the union of heteroclinic cycles, where in particular, for any pair of saddles in the network, there is a sequence of heteroclinic connections that links them. Heteroclinic cycles arise robustly in differential equations that are equivariant under the action of a group of symmetries as a connected component of the group orbit of a heteroclinic cycle. We refer the reader to Golubitsky and Stewart [13] for more information about heteroclinic cycles and symmetry in differential equations.

In the next section, we construct an example of a polynomial vector field whose organising centre has a heteroclinic network which originates a wide range of phenomena around it.

2. The vector field and a framework of the article

Our object of study is the two parameter family of vector fields $\mathbf{X}(X, \lambda_1, \lambda_2)$ on the unit sphere $\mathbf{S}^3 \subset \mathbf{R}^4$, defined for $X = (x_1, x_2, x_3, x_4) \in \mathbf{S}^3$ by the differential equation in $\mathbf{R}^4$:

$$\left\{ \begin{array}{l} \dot{x}_1 = x_1(1 - r^2) - x_2 - \alpha_1 x_1 x_4 + \alpha_2 x_1 x_4^2 + \lambda_2 x_3^2 x_4 \\ \dot{x}_2 = x_2(1 - r^2) + x_1 - \alpha_1 x_2 x_4 + \alpha_2 x_2 x_4^2 \\ \dot{x}_3 = x_3(1 - r^2) + \alpha_1 x_3 x_4 + \alpha_2 x_3 x_4^2 + \lambda_1 x_1 x_2 x_4 - \lambda_2 x_1 x_3 x_4 \\ \dot{x}_4 = x_4(1 - r^2) - \alpha_1(x_3^2 - x_1^2 - x_2^2) - \alpha_2 x_4(x_1^2 + x_2^2 + x_3^2) - \lambda_1 x_1 x_2 x_3 \end{array} \right. \tag{2.3}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $\alpha_2 < 0 < \alpha_1$ with $\alpha_1 + \alpha_2 > 0$.

Equation (2.3) with $\lambda_1 = \lambda_2 = 0$ was studied by Aguiar *et al* [3], we describe their results briefly. The unit sphere $\mathbf{S}^3$ is invariant under the flow of (2.3) and every trajectory with nonzero initial condition is asymptotic to it in forward time, so the vector fields $\mathbf{X}(X, \lambda_1, \lambda_2)$ are well

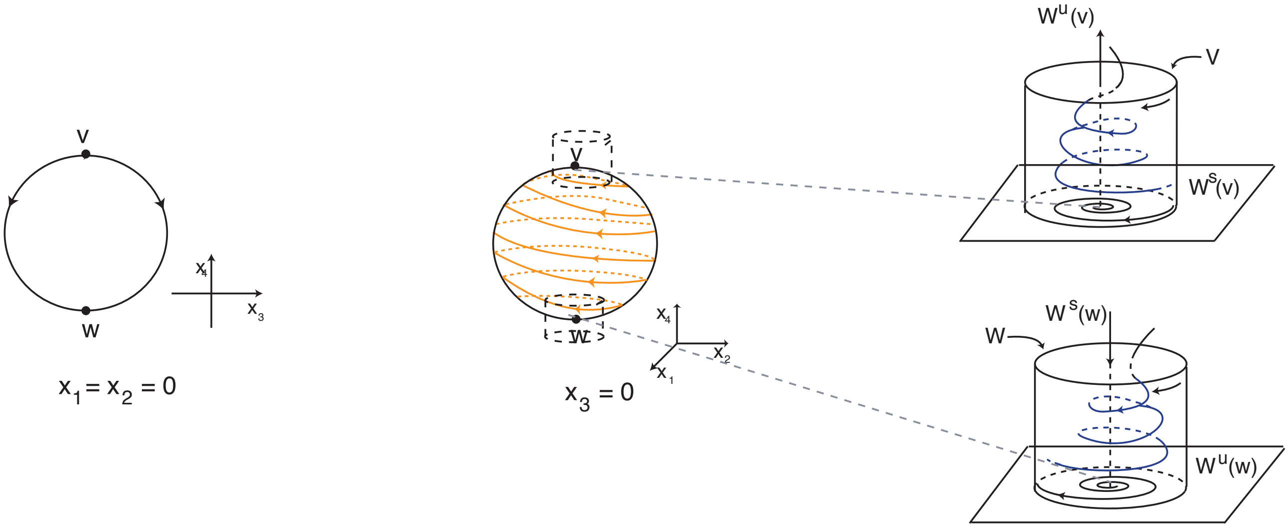

FIGURE 1. Dynamics of the organising centre $\mathbf{X}_0(X) = \mathbf{X}(X, 0, 0)$. Left: The one-dimensional heteroclinic connection from $\mathbf{v}$ to $\mathbf{w}$ on the invariant circle $Fix(\mathbf{SO(2)}(\gamma_1)) \cap \mathbf{S}^3$. Centre: The two-dimensional heteroclinic connection from $\mathbf{w}$ to $\mathbf{v}$ on the invariant two–sphere $Fix(\mathbf{Z}_2(\gamma_2)) \cap \mathbf{S}^3$. Right: Open neighbourhoods of $\mathbf{v}$ and $\mathbf{w}$, inside which the direction of solutions turning around the connection $[\mathbf{v} \to \mathbf{w}]$ is the same.

defined. There are two equilibria given by:

$$\mathbf{v} = (0, 0, 0, +1) \quad \text{and} \quad \mathbf{w} = (0, 0, 0, -1)$$

and the linearisation of $\mathbf{X}(X, \lambda_1, \lambda_2)$ at $(0, 0, 0, \varepsilon)$ with $\varepsilon = \pm 1$ has eigenvalues

$$\alpha_2 - \varepsilon\alpha_1 \pm i \quad \text{and} \quad \alpha_2 + \varepsilon\alpha_1.$$

Then, under the conditions above, $\mathbf{v}$ and $\mathbf{w}$ are hyperbolic saddle-foci, $\mathbf{v}$ has one-dimensional unstable manifold and two-dimensional stable manifold; $\mathbf{w}$ has one-dimensional stable manifold and two-dimensional unstable manifold.

The vector field $\mathbf{X}_0(X) = \mathbf{X}(X, 0, 0)$ is equivariant under the action of the compact Lie group $\mathbf{SO(2)} \oplus \mathbf{Z}_2(\gamma_2)$ where $\psi_\theta \in \mathbf{SO(2)}$ acts as:

$$\psi_\theta(x_1, x_2, x_3, x_4) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3, x_4)$$

and $\gamma_2 \in \mathbf{Z}_2(\gamma_2)$ acts as:

$$\gamma_2(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, x_4).$$

The one-dimensional invariant manifolds of $\mathbf{v}$ and $\mathbf{w}$ lie inside the invariant circle $Fix(\mathbf{SO(2)}) \cap \mathbf{S}^3$ and the two-dimensional invariant manifolds lie in the invariant two-sphere $Fix(\mathbf{Z}_2(\gamma_2)) \cap \mathbf{S}^3$. Thus, symmetry forces the invariant manifolds of $\mathbf{v}$ and $\mathbf{w}$ to be in a very special position: they coincide, see Figure 1. The two saddle-foci, together with their invariant manifolds form a heteroclinic network Σ that is asymptotically stable by the criteria of Krupa and Melbourne [20, 21]. Indeed since $\alpha_2 < 0 < \alpha_1$, it follows that:

$$\rho = \frac{C_\mathbf{v}}{E_\mathbf{v}} \frac{C_\mathbf{w}}{E_\mathbf{w}} = \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right)^2 > 1,$$

where E_p and C_p denote the real parts of the expanding and contracting eigenvalues of DX_0 at $\mathbf{v}$ and $\mathbf{w}$, respectively.

The network Σ can be decomposed into two cycles. Due to the symmetry and to the asymptotic stability, trajectories whose initial condition starts outside the invariant fixed point subspaces will approach in positive time one of the cycles. The fixed point hyperplanes prevent

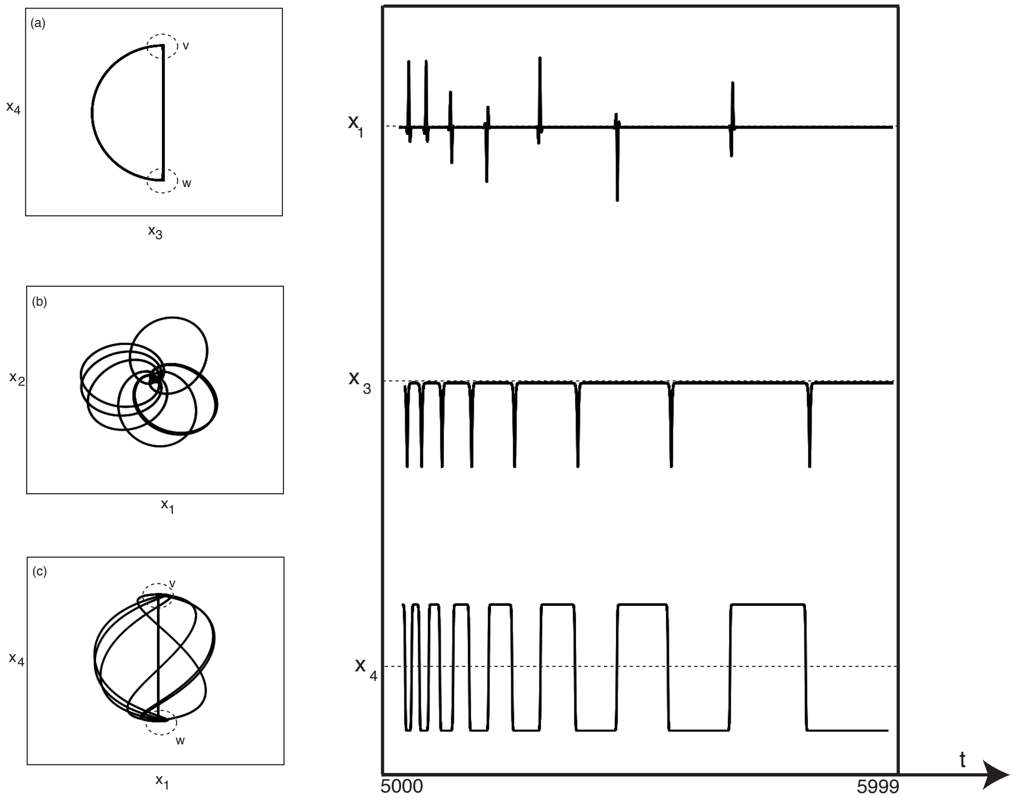

FIGURE 2. Left: Projection in the (x_3, x_4), (x_1, x_2) and (x_1, x_4)–planes of the trajectory with initial condition $(-0.5000, -0.1390, -0.8807, 0.3013)$ for the flow corresponding to $\mathbf{X}_0(X) = \mathbf{X}(X, 0, 0)$, with $\alpha_1 = 1$, $\alpha_2 = -0.1$. Right: Corresponding time series (same parameters and same initial condition). The simulations omit the initial transient (the independent variable t varies between 5000 and 5999).

random visits to the two cycles; a trajectory that approaches one of the cycles in Σ is shown in figure 2. The time series of the figure shows the increasing intervals of time spent near the equilibria. The sejourn time in any neighbourhood of one of saddle-foci increases geometrically with ratio ρ.

The symmetries of the organising centre are broken when either λ_1 or λ_2 is not zero. In section 3, after some additional results on $\mathbf{X}_0(X)$, we discuss briefly the dynamics of $\mathbf{X}(X, 0, \lambda_2)$, with $\lambda_2 \neq 0$, when the rotation symmetry $\mathbf{SO}(2)$ is broken, destroying the network Σ. Then we exploit the more dynamically interesting situation that arises for $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case the reflection symmetry $\mathbf{Z}_2$ is broken, as well as part of the $\mathbf{SO}(2)$–symmetry. The two-dimensional connection breaks into a pair of one-dimensional ones, but due to the remaining symmetry, the one-dimensional connection from $\mathbf{v}$ to $\mathbf{w}$ is preserved. This gives rise to two Bykov cycles – heteroclinic cycles with two saddle-foci of different Morse indices, in which the one-dimensional invariant manifolds coincide and the two dimensional invariant manifolds have an isolated transverse intersection. Analytical proof of transverse intersection of two invariant manifolds is usually difficult to obtain but this can be achieved in our example, and is done in the Appendix.

When both λ_1 and λ_2 are non zero, all the symmetry is broken, hence the one-dimensional connection from $\mathbf{v}$ to $\mathbf{w}$ disappears. Although the attracting heteroclinic cycles disappears, some nearby attracting structures remain, this is discussed in section 4. Our results and conjectures

are illustrated by numerical simulations, which have been obtained using the dynamical systems packages *Dstool* and *Matlab*.

3. Partial symmetry breaking

The existence of the attracting heteroclinic network Σ for $\mathbf{X}_0$ requires two separate symmetries to allow structurally stable connections within the two invariant fixed point subspaces. Throughout this paper, we control the degree to which the two symmetries are broken by two parameters, λ_1 and λ_2: λ_1 controls the magnitude of the $\mathbf{SO(2)} \times \mathbf{Z}_2(\gamma_2)$–symmetry-breaking term and λ_2 controls the breaking of the pure reflectional symmetry $\mathbf{Z}_2(\gamma_1)$. The parameters λ_1 and λ_2 play exactly the same role as in [17].

Before discussing the effects of breaking the symmetry in (2.3) when either λ_1 or λ_2 is non zero, we need some additional information on the fully symmetric case.

3.1. The organising centre.

For $\mathbf{X}_0$, there are two possibilities for the geometry of the flow around each saddle-focus of the network Σ, depending on the direction solutions turn around $[\mathbf{v} \to \mathbf{w}]$, as discussed in Labouriau and Rodrigues [17] . The next proposition shows that each solution when close to $\mathbf{v}$ turns in the same direction as when close to $\mathbf{w}$.

Proposition 1. *In $\mathbf{S}^3$, there are open neighbourhoods V and W of $\mathbf{v}$ and $\mathbf{w}$, respectively, such that, for any trajectory of $\mathbf{X}_0$ going from V to W, the direction of its turning around the connection $[\mathbf{v} \to \mathbf{w}]$ is the same in V and in W.*

Proof. The explicit expression of $\mathbf{X}_0$ in spherical coordinates is the special case $\lambda_1 = 0$ of equation (A.6) in the appendix. The equation for the angular coordinate φ in the plane $(x_1, x_2, 0, 0)$ is $\dot{\varphi} = 1$. Since this plane is perpendicular to $Fix(\mathbf{SO(2)}(\gamma_1))$, where the connection $[\mathbf{v} \to \mathbf{w}]$ lies, trajectories must turn around the connection in the same direction. □

The property in Proposition 1 is persistent under isotopies: if it holds for the organising centre $\lambda_1 = \lambda_2 = 0$, then it is still valid in smooth one-parameter families containing it, as long as there is still a connection. In particular, property (P8) of Labouriau and Rodrigues [17] is verified and thus their results may be applied to the present work.

3.2. Breaking the two-dimensional heteroclinic connection.

For $\lambda_1 \neq 0$, the vector field $\mathbf{X}_1(X) = \mathbf{X}(X, \lambda_1, 0)$ is no longer $\mathbf{SO(2)} \oplus \mathbf{Z}_2(\gamma_2)$ but is still equivariant under the action of $\gamma_1 := \psi_\pi \in \mathbf{SO(2)}$. This breaks the two-dimensional connection $[\mathbf{w} \to \mathbf{v}]$ into a transverse intersection of invariant manifolds.

Theorem 2. *The vector field $\mathbf{X}_1(X) = \mathbf{X}(X, \lambda_1, 0)$ on $\mathbf{S}^3$ has symmetry group $\mathbf{Z}_2(\gamma_1)$ and for small $\lambda_1 \neq 0$ it has a heteroclinic network $\Sigma^\star$ involving the two equilibria $\mathbf{v}$ and $\mathbf{w}$ with the following properties:*

1. *there are two one-dimensional heteroclinic connections from $\mathbf{v}$ to $\mathbf{w}$ inside $Fix(\mathbf{Z}_2(\gamma_1)) \cap \mathbf{S}^3$;*
2. *there are no homoclinic connections to the equilibria;*
3. *the two-dimensional invariant manifolds of $\mathbf{v}$ and $\mathbf{w}$ intersect transversely along one-dimensional connections from $\mathbf{w}$ to $\mathbf{v}$.*

Moreover the connections from $\mathbf{v}$ to $\mathbf{w}$ persist under small perturbations that preserve the γ_1–symmetry.

The new network $\Sigma^\star$ is qualitatively different from Σ. The transversality of $W^u(\mathbf{w})$ and $W^s(\mathbf{v})$ and the persistence of the heteroclinic connections from $\mathbf{v}$ to $\mathbf{w}$ give rise to a *Bykov cycle.* In the non-symmetric context, Bykov cycles arise as bifurcations of codimension 2, the points in parameter space where they appear are called *T-points* in the literature – see figure 3. The existence of a Bykov cycle implies the existence of a bigger network: beyond the original

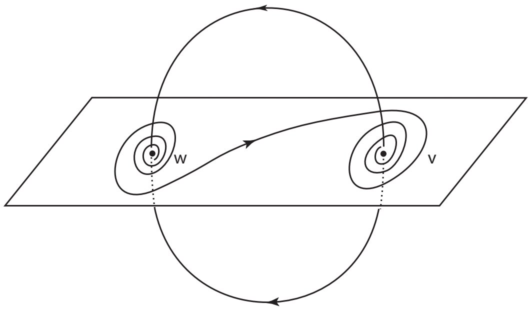

FIGURE 3. Bykov cycle $\Sigma^\star$: heteroclinic cycle associated to two saddle-foci of different Morse indices, in which the one-dimensional invariant manifolds coincide and the two-dimensional invariant manifolds intersect transverselly.

transverse connections, there exists infinitely many subsidiary heteroclinic connections turning around the original Bykov cycle.

Proof. Clearly the perturbation term breaks all the symmetries of $\mathbf{X}_1(X)$ except γ_1. It is also immediate that $\gamma_1(\mathbf{v}) = \mathbf{v}$, $\gamma_1(\mathbf{w}) = \mathbf{w}$. The circle $Fix(\mathbf{Z}_2(\gamma_1)) \cap \mathbf{S}^3$ is still flow-invariant and contains no other equilibria, thus it still consists of the two equilibria $\mathbf{v}$ and $\mathbf{w}$ and the two connections from $\mathbf{v}$ to $\mathbf{w}$. The plane $Fix(\mathbf{Z}_2(\gamma_1))$ will remain invariant under any perturbation that preserves the symmetry. Since on this plane $\mathbf{v}$ is a saddle and $\mathbf{w}$ is a sink, the connection persists under small perturbations. Since $\mathbf{v}$ and $\mathbf{w}$ are hyperbolic, $Fix(\mathbf{Z}_2(\gamma_1)) \supset [\mathbf{v} \to \mathbf{w}]$, and $\dim W^u(\mathbf{v}) = \dim W^u(\mathbf{w}) = 1$, item (2) follows. Breaking the $\mathbf{Z}_2(\gamma_2)$–equivariance is necessary for the existence of transverse intersection of the manifolds $W^u(\mathbf{w})$ and $W^s(\mathbf{v})$. Nevertheless, the manifolds could intersect non transversely. The proof of transversality of the intersection of the two-dimensional manifolds, using the Melnikov method is similar to that given in Aguiar *et al* [3]. For completeness, we give the proof in Appendix (A.2). □

When $\lambda_1 \neq 0$, we observe an explosion of suspended horseshoes in the neighbourhood of $\Sigma^\star$, a phenomenon often called *instant chaos.*

Theorem 3. *For generic $\alpha_1 \neq \alpha_2$ and for any small $\lambda_1 \neq 0$, the vector field $\mathbf{X}_1 = \mathbf{X}(X, \lambda_1, 0)$ is $\mathbf{Z}_2(\gamma_1)$–equivariant and has a compact spiralling set $\Lambda \subset \mathbf{S}^3$ containing the following:*

(1) *a heteroclinic network $\Sigma^\star$ involving the two saddle-foci $\mathbf{v}$ and $\mathbf{w}$ with a transverse intersection of the two dimensional invariant manifolds;*
(2) *a uniformly hyperbolic compact set $\mathcal{H} = \bigcup_{i \in \mathbf{Z}} H_i$, where $\{H_i\}_{i \in \mathbf{Z}}$ is an increasing sequence of invariant sets, accumulating on $\Sigma^\star$. Each H_i is topologically conjugate to the suspension of a full shift over a finite set of symbols.*

Proof. The proof of (1) follows from Theorem 2. Note that transversality implies the existence of a larger network with infinitely many heteroclinic connections.

For $\lambda_1 = 0$, the heteroclinic network Σ for $\mathbf{X}_0$ is asymptotically stable. In particular, there are arbitrarily small compact neighbourhoods of Σ such that the vector field is transverse to their boundaries where it points inwards. Let $\mathcal{N}$ be one of these neighbourhoods. There exists $\lambda_\star > 0$ such that for $\lambda_1 < \lambda_\star$, the vector field $\mathbf{X}_1$ is still transverse to the boundary of $\mathcal{N}$ and $\Sigma^\star \subset \mathcal{N}$. Thus $\mathcal{N}$ is a compact set that is positively invariant under the flow of $\mathbf{X}_1$. Hence $\mathcal{N}$ contains an attractor.

If $\alpha_1 \neq \alpha_2$, using Samovol [30] the flow of $\mathbf{X}_1$ is C^1–linearisable around each saddle-focus. Linearisation may fail under resonance conditions that correspond to curves in the (α_1, α_2)–plane. This restriction has zero Lebesgue measure, and thus it does not place serious constraint on our analysis. We use the basic ideas of Aguiar *et al* [4] and Aguiar *et al* [5] adapted to our

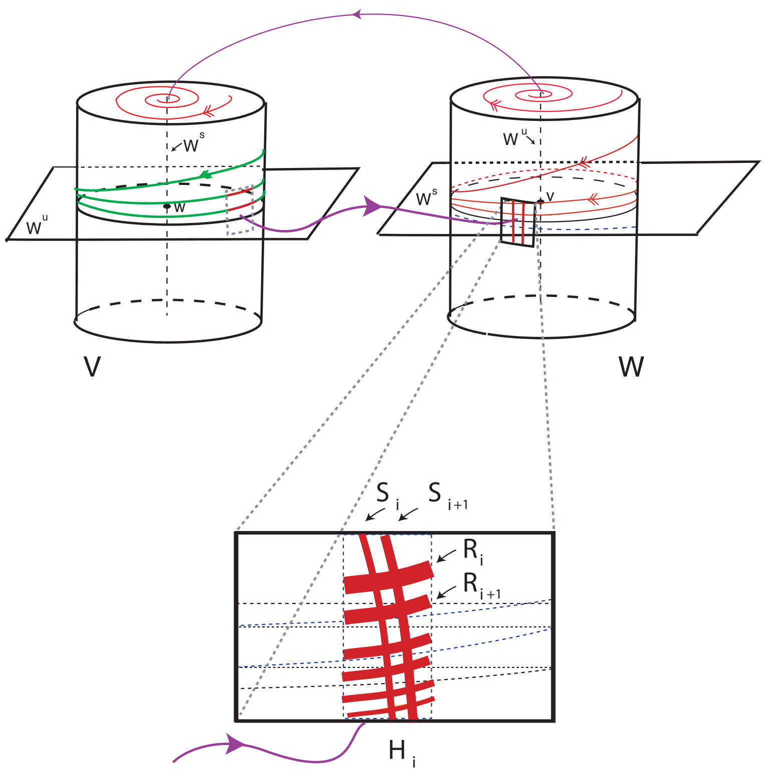

FIGURE 4. Construction of the suspended horseshoe $\mathcal{H}_i$ for $\mathbf{X}_1$ near the cycle, giving rise to the Cantor set on the wall of the cylinder, generated by an arbitrarily large number of strips.

purposes to show the existence of the suspended horseshoe $\mathcal{H}$ accumulating on the heteroclinic network (see figure 4). Inside each open set where the linearisation has been done, we take a cylindrical neighbourhood of each equilibrium, V and W.

The boundary of each cylinder forms an *isolating block*: the flow is transverse to the cylinder walls, top and bottom. Near the saddle-focus $\mathbf{v}$, the flow goes in at the cylinder walls, and it goes out at the top and bottom. Near $\mathbf{w}$, the flow goes in at the cylinder top and bottom and it goes out at the wall. Inside the cylinder the vector field is linear, so the transition from the wall to top/bottom and from top/bottom to the wall is well understood. The transition between cylinders takes place in a flow-box fashion around each connection.

Any segment of initial conditions lying across the stable manifold of $\mathbf{v}$ is wrapped by the flow around the isolating block. By the λ–lemma it accumulates as a spiral on the unstable manifold of $\mathbf{v}$ and then on the stable manifold of the next saddle, $\mathbf{w}$ – see figure 4. This spiral of initial conditions on the top/bottom of the neighbourhood around $\mathbf{w}$ is mapped, by the local map near $\mathbf{w}$, into points lying on a helix accumulating on its unstable manifold, which crosses transversely the local stable manifold of $\mathbf{v}$ infinitely many times.

The composition of consecutive local maps and transition functions, transforms the original segment of initial conditions lying across $W^s(\mathbf{v})$ into infinitely many segments with the same property. Thus, it is possible to construct recursively a nested sequence of strips accumulating on the stable manifold of $\mathbf{v}$ that follow prescribed sequences of heteroclinic connections and that return to the cylindrical neighbourhood of $\mathbf{v}$. Using now Proposition 3 and Corollary 5 of Aguiar *et al* [4] adapted to our purposes, instead of a nested sequence of intervals, we may construct, as in figure 4, a sequence of rectangles $\mathcal{R}_i$, accumulating on $\Sigma^\star$, such that the forward iterate of initial conditions in $\mathcal{R}_i$ lie in strips $\mathcal{S}_i$ that cross transversely all the $\mathcal{R}_i$, giving rise to a chain of suspended horseshoes $\mathcal{H}_i$. □

The suspended horseshoe $\mathcal{H}$ containing the heteroclinic network on its closure is semi-conjugate to a full shift over an infinite alphabet with countably many symbols. The overall dynamics is organised around the suspended horseshoes. In [4], it is shown that that if Π is a transverse section to the suspended horseshoe, the first return map of $\mathcal{H}$ to Π is hyperbolic at all points where the horseshoe is well defined. Points lying on the invariant manifolds of $\mathbf{v}$ and $\mathbf{w}$ are dense in $\mathcal{H}$. This chain of horseshoes might have positive Lebesgue measure like the "fat Bowen horseshoe". Rodrigues [27] proved that the shift dynamics does not trap most trajectories in any small neighbourhood of $\Sigma^\star$:

Corollary 4. *Let $N^{\Sigma^\star}$ be a tubular neighbourhood of one of the Bykov cycles $\Sigma^\star$. Then in any cross-section $S_q \subset N^{\Sigma^\star}$ at a point q in $[\mathbf{w} \to \mathbf{v}]$ the set of initial conditions in $S_q \cap N^{\Sigma^\star}$ that do not leave $N^{\Sigma^\star}$ for all time has zero Lebesgue measure.*

3.2.1. *Heteroclinic Switching and Subsidiary Dynamics.* The overall dynamics is organised around the suspended horseshoes. Given a cross section Π to $\Sigma^\star$, in the restriction to an uniformly hyperbolic invariant compact set, the dynamics can be conjugated to a full shift over a finite alphabet. In particular, the *topological entropy* of the corresponding flow is positive. One astonishing property is the possibility of shadowing the heteroclinic network $\Sigma^\star$ by the property called *heteroclinic switching*: any infinite sequence of pseudo-orbits defined by admissible heteroclinic connections can be shadowed, as we proceed to define.

For the heteroclinic network $\Sigma^\star$ with node set $\{\mathbf{v}, \mathbf{w}\}$, a *path of order k* on $\Sigma^\star$ is a finite sequence $s^k = (c_j)_{j\in\{1,\ldots,k\}}$ of heteroclinic connections $c_j = [A_j \to B_j]$ in $\Sigma^\star$ such that $A_j, B_j \in \{\mathbf{v}, \mathbf{w}\}$ and $B_j = A_{j+1}$. An infinite path corresponds to an infinite sequence of connections in $\Sigma^\star$. Let $N_{\Sigma^\star}$ be a neighbourhood of the network $\Sigma^\star$ and let U_A be a neighbourhood of each node A. For each heteroclinic connection in $\Sigma^\star$, consider a point p on it and a small neighbourhood V of p. We assume that the neighbourhoods of the nodes are pairwise disjoint, as well for those of points in connections.

Given neighbourhoods as before, the trajectory $\varphi_f(t,q)$, *follows* the finite path $s^k = (c_j)_{j\in\{1,\ldots,k\}}$ of order k, if there exist two monotonically increasing sequences of times $(t_i)_{i\in\{1,\ldots,k+1\}}$ and $(z_i)_{i\in\{1,\ldots,k\}}$ such that for all $i \in \{1,\ldots,k\}$, we have $t_i < z_i < t_{i+1}$ and:

1. $\varphi_f(t,q) \subset N_{\Sigma^\star}$ for all $t \in (t_1, t_{k+1})$;
2. $\varphi_f(t_i,q) \in U_{A_i}$ and $\varphi(z_i,q) \in V_i$ and
3. for all $t \in (z_i, z_{i+1})$, $\varphi_f(t,q)$ does not visit the neighbourhood of any other node except that of A_{i+1}.

There is *finite switching* near $\Sigma^\star$ if for each finite path there is a trajectory that follows it. Analogously, we define *infinite switching* near $\Sigma^\star$ if every forward infinite sequence of connections in the network is shadowed by nearby trajectories. Infinite switching near $\Sigma^\star$ follows from the proof of Theorem 3 and the results of Aguiar *et al* [5]. The solutions that realise switching lie in a tubular neighbourhood $N^{\Sigma^\star}$ of $\Sigma^\star$, hence from the results of Rodrigues [27], it follows that:

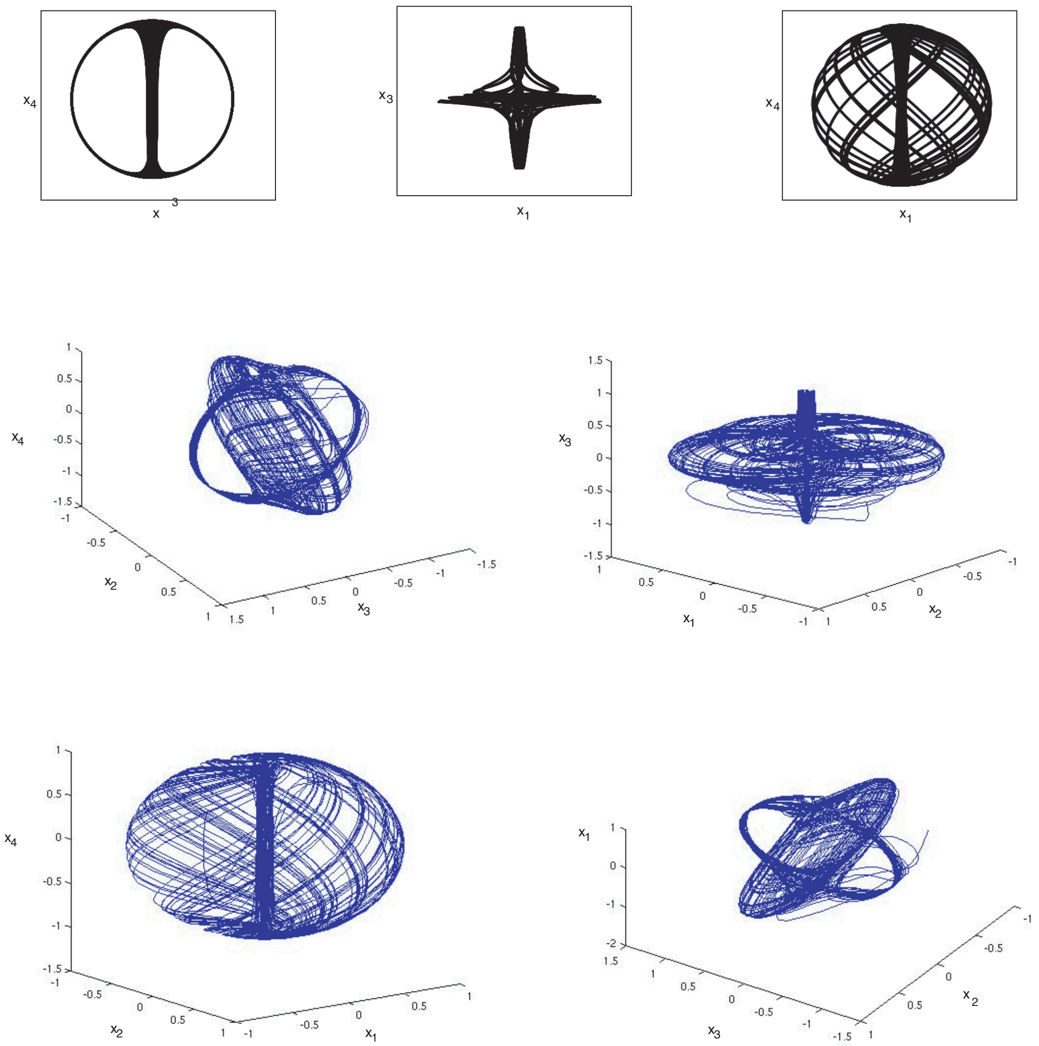

FIGURE 5. The spiralling set Λ. Top: Projection in the (x_3, x_4), (x_1, x_3) and (x_1, x_4)–planes of the trajectory with initial condition $(-0.5000, -0.1390, -0.8807, 0.3013)$ for the flow of $\mathbf{X}_1$, with $\alpha_1 = 1$, $\alpha_2 = -0.1$ and $\lambda_1 = 0.05$. Centre and Bottom: Projection in the (x_2, x_3, x_4), (x_1, x_2, x_3), (x_1, x_2, x_4) and (x_1, x_3, x_4)–hyperplanes of the trajectory with the same initial condition and parameters.

Corollary 5. *There is infinite switching near the network $\Sigma^\star$. Infinite switching is realized by a set of initial conditions with zero Lebesgue measure, whereas finite switching occurs for a set of infinite conditions with positive Lebesgue measure.*

The complex eigenvalues force the spreading of solutions around the unstable manifold of $\mathbf{w}$, allowing visits to all possible connections starting at $\mathbf{w}$. The transversality enables the existence of solutions that follow heteroclinic connections on the two different connected components of $\mathbf{S}^3 \backslash W^s_{loc}(\mathbf{v})$, the upper and the lower part on the wall of the cylinder – see figure 4.

From the results of [17], it also follows that near $\Sigma^\star$ the only heteroclinic connections from $\mathbf{v}$ to $\mathbf{w}$ are the original ones, that there are no homoclinic connections associated to the equilibria and that the finite switching near $\Sigma^\star$ may be realised by an n–pulse heteroclinic connection

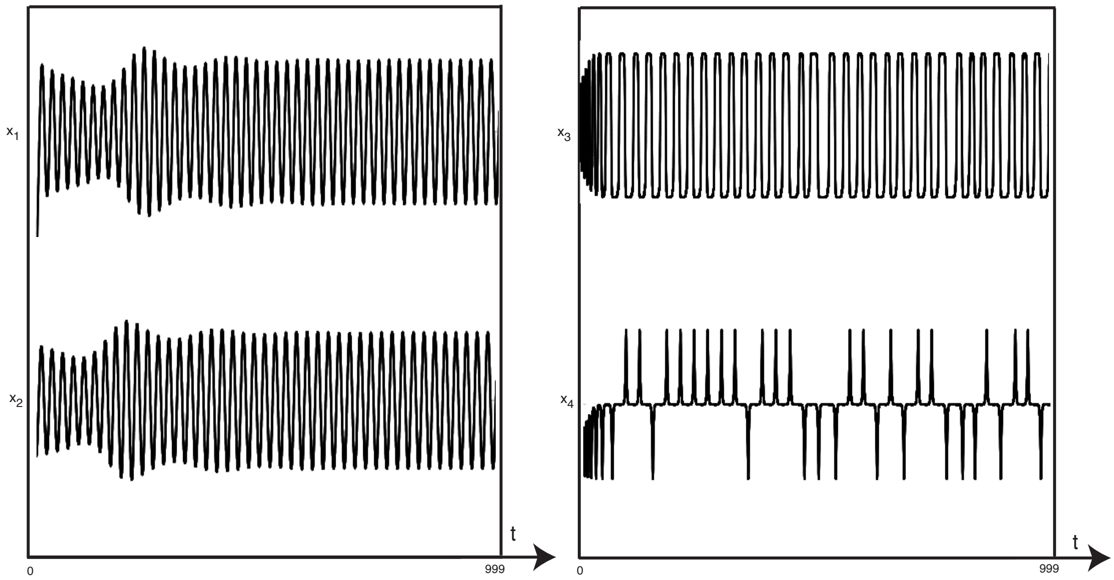

FIGURE 6. Time series for the trajectory with initial condition $(-0.5000, -0.1390, -0.8807, 0.3013)$ for the flow of $\mathbf{X}_1$, with $\alpha_1 = 1$, $\alpha_2 = -0.1$ and $\lambda_1 = 0.05$.

$[\mathbf{w} \to \mathbf{v}]$. Moreover, among all the solutions which appear in $\mathcal{H}$, there are infinitely many knot types, inducing all link types.

We illustrate the chaotic behaviour in the projeted phase portraits of figure 5 and in the time series of figure 6 corresponding to a trajectory that stays near the heteroclinic network. Observing figure 5, it is clear why we call the nonwandering set Λ of $\mathbf{X}_1$ a *spiralling set.* The time series of figure 6 also suggest heteroclinic switching: the trajectory follows a sequence of heteroclinic connections in a random order. We stress that this can occur because the symmetries have been broken.

Under generic perturbations (not necessarily equivariant), there is still an invariant topological sphere, since $\mathbf{S}^3$ is normally hyperbolic, and the transverse connections are preserved. The spiralling set presented in Theorem 3 may lose branches, changing its nature. Any compact flow-invariant set near the ghost of the heteroclinic cycle, disjoint from it, contains an uniformly hyperbolic basic set displaying structurally stable homoclinic classes in its unfolding, where the results of Rodrigues *et al* [28] may be applied. We will return to this issue in section 4 below.

3.3. **Nonhyperbolic behaviour.** Based in [18], applying the construction of the proof of Theorem 3 to the unstable manifold of $\mathbf{w}$, we obtain the following result:

Theorem 6. *There are values $\lambda_1^\star$ arbitrarily close to 0, for which the flow of $\mathbf{X}(X, \lambda_1^\star, 0)$ has a heteroclinic tangency between $W^u(\mathbf{w})$ and $W^s(\mathbf{v})$, coexisting with the transverse connections in $\Sigma^\star$.*

The tangencies of Theorem 6 take place far from the invariant set $\mathcal{H}$ of Theorem 3. Newhouse results [25, 26] ensure the existence of infinitely many sinks nearby.

Proof. First of all, we define some terminology. As in the proof of Theorem 3 (see figure 10), let V and W be cylindrical neighbourhoods of each equilibrium point after linearising $\mathbf{X}$ around them. Let

- $O_{\mathbf{v}}^{out}$ be the top/bottom of the cylinder V, where the flow goes out of V;
- $I_{\mathbf{v}}^{in}$ be the wall of the cylinder V, where the flow goes in V;

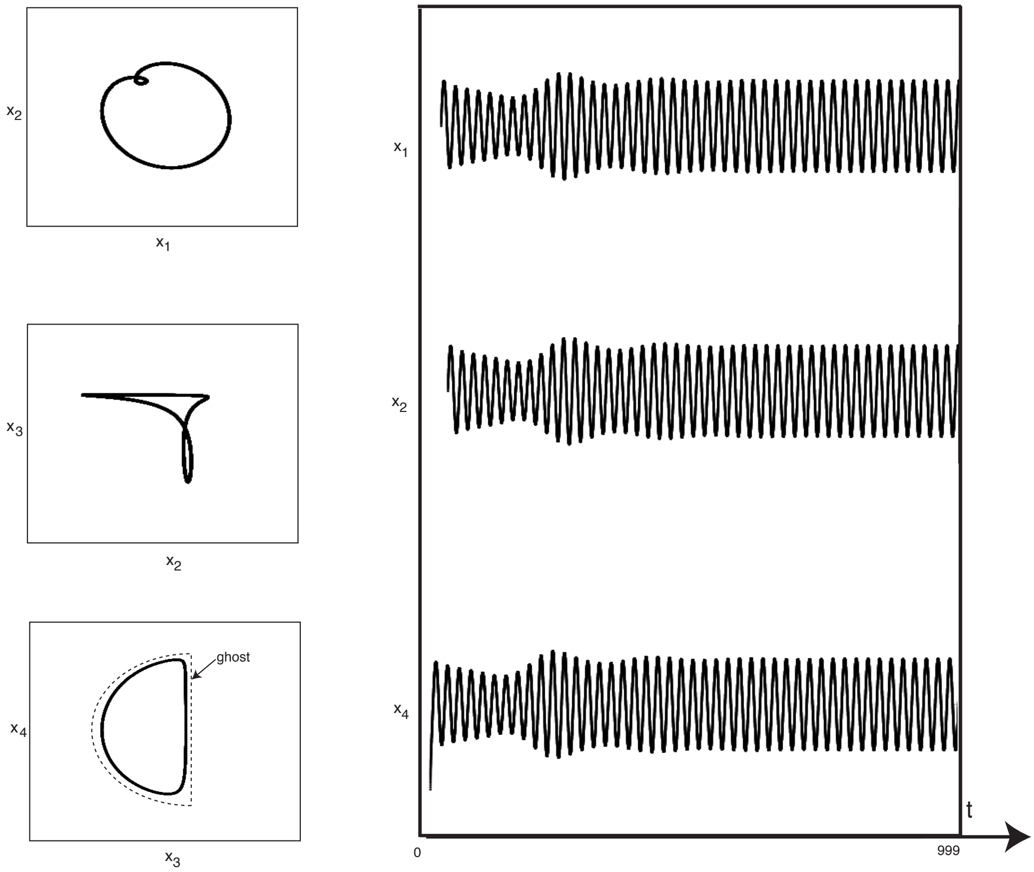

FIGURE 7. Left: Projection in the (x_1, x_2), (x_2, x_3) and (x_3, x_4)–planes of the trajectory with initial condition $(-0.5000, -0.1390, -0.8807, 0.3013)$ for the flow corresponding to the equation (2.3), with $\lambda_1 = 0$, $\lambda_2 = 0.05$ for $\alpha_1 = 1$ and $\alpha_2 = -0.1$. The dotted line on the (x_3, x_4)–plane indicates the position of the original cycle. Right: Time series for the corresponding trajectory (same initial condition and same parameters).

- $O^{out}_{\mathbf{w}}$ be the wall of the cylinder W, where the flow goes out of W;
- $I^{in}_{\mathbf{w}}$ be the top/bottom of the cylinder W, where the flow goes in W.

Consider the closed curve in where $W^u_{loc}(\mathbf{w})$ intersects $In(\mathbf{v})$, as shown in figure 10. For small $\lambda_1 \neq 0$, this curve has a point of maximum height that divides it in two components, each one of which is mapped into a helix around $Out(\mathbf{w})$. The two helices taken together form a curve with a fold point (see Figure 8). Varying λ_1 moves the fold point around $Out(\mathbf{w})$, exponentially fast in λ_1. As λ_1 tends to zero, the fold point will cross $W^s_{loc}(\mathbf{v}) \cap Out(\mathbf{w})$ infinitely many times, creating heteroclinic tangencies.

□

3.4. **Breaking the one-dimensional heteroclinic connection.** For $\lambda_2 \neq 0$, the vector field $\mathbf{X}_2 = \mathbf{X}(X, 0, \lambda_2)$ is no longer $\mathbf{SO(2)}$–equivariant but is still equivariant under the action of γ_2. Since the heteroclinic connections $[\mathbf{v} \to \mathbf{w}]$ lie on $Fix(\mathbf{Z}_2(\gamma_1)) = \{(0, 0, x_3, x_4), x_3, x_4 \in \mathbf{R}\}$, these connections disappear, destroying the heteroclinic network Σ, but for small values of λ_2 there will still be an attractor lying close to the original cycle. Theorem 6 of [17] shows that for sufficienly small $\lambda_2 \neq 0$, each heteroclinic cycle that occurred in the fully symmetric case is replaced by a stable and hyperbolic periodic solution. Using the reflection symmetry γ_2,

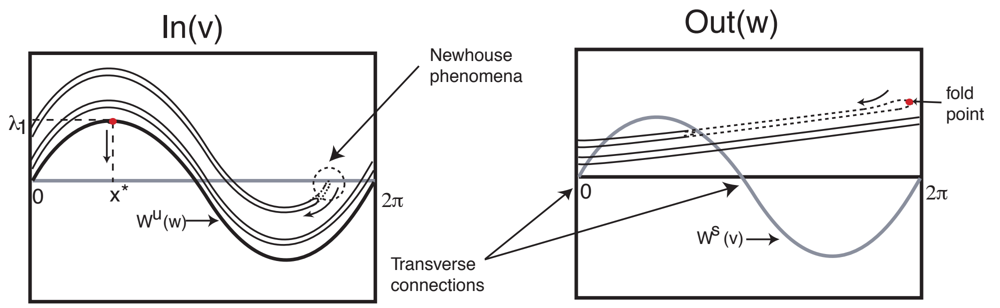

FIGURE 8. Heteroclinic tangency between $W^u(\mathbf{w})$ and $W^s(\mathbf{v})$, coexisting with the transverse connections in $\Sigma^\star$.

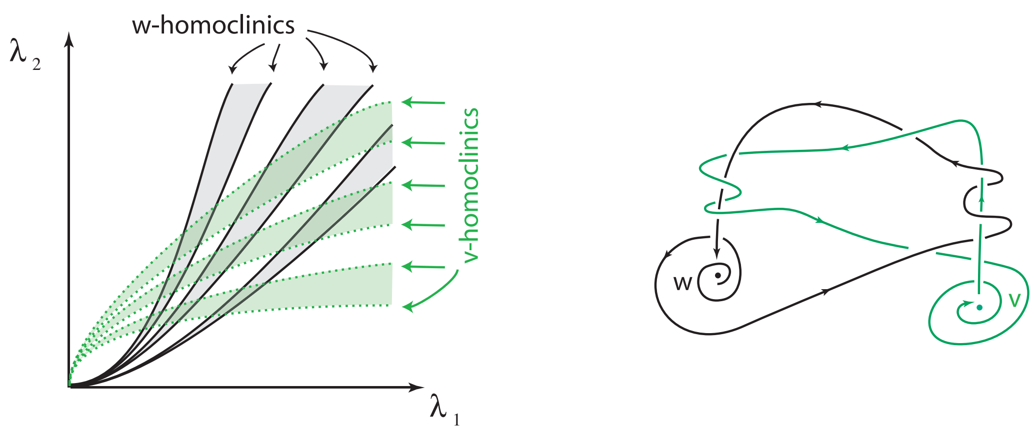

FIGURE 9. Left: Qualitative bifurcation diagram for the family $\mathbf{X}(X, \lambda_1, \lambda_2)$. At the solid curves tangent to the λ_1 axis there is an expanding Shilnikov homoclinic at $\mathbf{w}$. Consecutive pairs of these curves bound a tongue (shaded) where $\mathbf{X}$ has infinitely many unstable periodic solutions associated to a suspended horseshoe. Contracting Shilnikov homoclinics at $\mathbf{v}$ appear at the dotted curves tangent to the λ_2 axis, that limit tongues where $\mathbf{X}$ has an attracting period orbit. Right: Two linked Shilnikov homoclinic loops appear when the two types of curve meet.

two stable periodic solutions co-exist, one in each connected component of $\mathbf{S}^3\backslash Fix(\mathbf{Z}_2(\gamma_2))$. Their period tends to $+\infty$ when $\lambda_2 \to 0$ and their basin of attraction must contain the basin of attraction of Σ. For sufficiently large λ_2, saddle-node bifurcations of these two periodic solutions occur. Figure 7 illustrates the existence of a single attracting periodic solution in each connected component $\mathbf{S}^3\backslash Fix(\mathbf{Z}_2(\gamma_2))$ of the phase space.

4. Breaking all the symmetry

4.1. **Linked homoclinic cycles.** In this section, we describe the global bifurcations that are most important for our analysis, which appear when both λ_1 and λ_2 are non-zero and $\mathbf{X}$ has no symmetry. Different chaotic dynamics in (2.3) organise a complex network of bifurcations involving rotating nodes, that had not yet been considered. The next theorem describes the bifurcation diagram for $\mathbf{X}$, shown in Figure 9.

Theorem 7. *The germ at the origin of the bifurcation diagram on the (λ_1, λ_2)–plane for $\mathbf{X}(X, \lambda_1\lambda_2)$ when $\lambda_1\lambda_2 \neq 0$ satisfies:*

(1) *For each (λ_1, λ_2), there is a suspended uniformly hyperbolic horseshoe, topologically conjugate to a full shift over a finite number of symbols. As (λ_1, λ_2) tends to $(\lambda_1, 0)$ with $\lambda_1 \neq 0$, the horseshoe accumulates on the set $\mathcal{H}$ of Theorem 3.*
(2) *There exists a countable family of open tongues, tangent at $(0,0)$ to the λ_2–axis, where $\mathbf{X}$ has an attracting periodic orbit. The closures of these tongues are pairwise disjoint. At the two curves comprising the boundary of each tongue there is an attracting Shilnikov homoclinic connection at $\mathbf{v}$.*
(3) *There exists a countable family of open tongues, tangent at $(0,0)$ to the λ_1–axis, where $\mathbf{X}$ exhibits a suspended nonhyperbolic horseshoe, topologically conjugate to a full shift over an finite set of symbols. The closures of these tongues are pairwise disjoint. At the two curves comprising the boundary of each tongue there is an unstable Shilnikov homoclinic connection at $\mathbf{w}$.*
(4) *The two sets of curves in the parameter plane described in (2) and (3) meet at isolated points where the two types of homoclinic connections coexist. These codimension two points accumulate at the origin. The closure of the pair of homoclinic orbits forms a link whose linking number tends to infinity as the points approach the origin of the (λ_1, λ_2)–plane.*

Theorem 7 follows from Theorems 7 and 8 in [17]. We sketch here an alternative proof, emphasising the transition between the two different types of chaos and their geometry.

Proof. We use the sets $I^{in}_{\mathbf{v}}$, $O^{out}_{\mathbf{v}}$, $I^{in}_{\mathbf{w}}$ and $O^{out}_{\mathbf{w}}$, defined in the proof of Theorem 6. In Theorem 3, for $\lambda_2 = 0$, we have found a nested chain of suspended horseshoes $\mathcal{H}$ contained in the spiralling set Λ. When generic perturbation terms are added, the hyperbolic horseshoes $\mathcal{H}$ lose infinitely many branches, only a finite number of strips survive, as stated in assertion (1). When $\lambda_1 \neq 0$ and $\lambda_2 = 0$, a segment s of initial conditions in $I^{in}_{\mathbf{v}}$, with s transverse to $W^s(\mathbf{v})$, is mapped into a spiral in $I^{in}_{\mathbf{w}}$ and then into a helix in $O^{out}_{\mathbf{w}}$ accumulating on $W^u(\mathbf{w})$. When $\lambda_2 \neq 0$, the spirals in $I^{in}_{\mathbf{w}}$ are off-centred (case (a) of figure 11) and will turn a finite number of times around $W^s(\mathbf{w})$. This is why the suspended horseshoes in assertion (1) have a finite number of strips. Suspended horseshoes arise when $W^u(\mathbf{v})$ is close to $W^s(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$, the number of branches increases when $W^u(\mathbf{v})$ approaches $W^s(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$.

From the coincidence of the invariant two dimensional manifolds near the equilibria in the fully symmetric case, we expect that when λ_1 is close to zero, $W^s(\mathbf{v})$ intersects the wall $O^{out}_{\mathbf{w}}$ of the cylinder W in a closed curve as in figure 10. When both symmetries are broken, the Bykov cycle $\Sigma^\star$ is destroyed, giving rise to Shilnikov homoclinic cycles involving the saddle-foci $\mathbf{v}$ and $\mathbf{w}$. The equilibria have different Morse indices, thus the dynamics near each homoclinic cycle is qualitatively different, see Shilnikov [31, 32]: all the homoclinic orbits associated to $\mathbf{v}$ are attracting since the contracting eigenvalue is larger than the expanding one; Each homoclinic cycle associated to $\mathbf{w}$ has a suspended horseshoe near it and thus infinitely many periodic solutions of saddle type occur in every neighbourhood of the homoclinicity of $\mathbf{w}$.

The existence of homoclinic orbits is not a robust property, they occur along the curves in the (λ_1, λ_2) plane described in assertions (2) and (3) – details on these curves are given in [17]. At values of (λ_1, λ_2) where the two types of curves cross, as in assertion (4), the homoclinics at $\mathbf{v}$ and $\mathbf{w}$ occur at different points in phase space.

Case (b) of figure 11 describes the homoclinic cycle at $\mathbf{w}$: initially $W^u(\mathbf{w})$ meets $I^{in}_{\mathbf{v}}$ at a closed curve, that is then mapped into a double spiral in $I^{in}_{\mathbf{w}}$ centred at $W^u(\mathbf{v}) \cap I^{in}_{\mathbf{w}}$. When this spiral meets $W^s(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$, an unstable homoclinic cycle associated to $\mathbf{w}$ is created. Similarly, the backwards image of $W^u(\mathbf{v})$ in $O^{out}_{\mathbf{w}}$ is a closed curve, that iterates back into a double spiral in $I^{in}_{\mathbf{w}}$ centred at $W^s(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$, creating an asymptotically stable homoclinic orbit of $\mathbf{v}$ when this spiral meets $W^u(\mathbf{v}) \cap I^{in}_{\mathbf{w}}$. The values of (λ_1, λ_2) where these intersections happen correspond to the two types of curves in figure 9. The double spiral $W^u(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$ is mapped into a spiral in

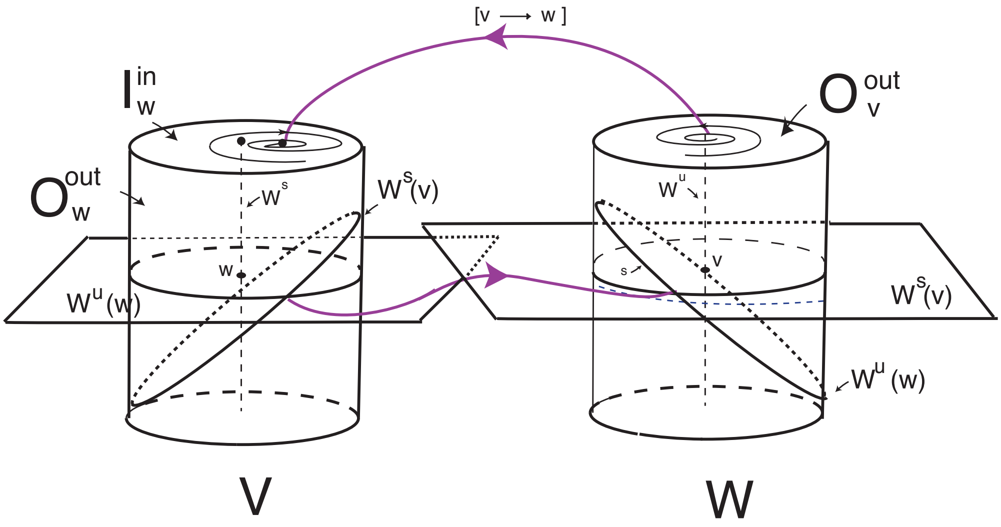

FIGURE 10. Conventions in the cylindrical neighbourhoods of $\mathbf{v}$ and $\mathbf{w}$ used in Theorem 7. Both $W^s_{loc}(\mathbf{v}) \cap O^{out}_{\mathbf{w}}$ and $W^u_{loc}(\mathbf{w}) \cap O^{in}_{\mathbf{v}}$ are closed curves in the boundaries of the cylinders for small values of $\lambda_1 \neq 0$.

$O^{out}_{\mathbf{w}}$ as in figure 11 (c). As one of the spiral arms gets closer to $W^s(\mathbf{w})$ in $I^{in}_{\mathbf{w}}$, its image winds around $O^{out}_{\mathbf{w}}$.

Along the curve where one of the arms of the double spiral in $W^u(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$ touches $W^s(\mathbf{w})$ in $I^{in}_{\mathbf{w}}$, there are some parameter values where the centre of the spiral in $O^{out}_{\mathbf{w}}$ meets the closed curve $W^s(\mathbf{v}) \cap O^{out}_{\mathbf{w}}$ (figure 11 (c)). At these points both homoclinic cycles coexist in different regions of the phase space and the basin of attraction of the homoclinic at $\mathbf{v}$ lies inside the gaps of the suspended horseshoes that appear and disappear from the homoclinicity at $\mathbf{w}$. □

We observe a mixing of regular and chaotic dynamics, in regions of parameter space (λ_1, λ_2) where the two types of tongues overlap. Examples of chaotic behaviour of this type should be regarded as quasiattractors due to the presence of stable periodic orbits within them. A lot more needs to be done before these transitions are well understood.

4.2. **Finite heteroclinic switching.** Generic breaking of the γ_1–symmetry destroys the heteroclinic network, because the two connections $[\mathbf{v} \to \mathbf{w}]$ break. Nevetheless, finite switching might be observed: for small values of λ_2 there are trajectories that visit neighbourhoods of finite sequences of nodes. This is because the spirals on top of the cylinder around $\mathbf{w}$ are off-centred and will turn a finite number of times around $W^s(\mathbf{w})$. In this way we may obtain points whose trajectories follow short finite paths on the network. As $W^u(\mathbf{v})$ gets closer to $W^s(\mathbf{w})$ (as the system moves closer to symmetry) the paths that can be shadowed get longer.

5. DISCUSSION

The goal of this paper is to construct explicitly a two-parameter polynomial differential equation, in which each parameter controls a type of symmetry breaking. We also prove analytically the transverse intersection of two invariant manifolds using the adapted Melnikov method which is very difficult in general. Some dynamical properties follow applying results in the literature. We also describe the transition between the different types of dynamics. As far as we know, our construction is new.

For three-dimensional flows, one homoclinic cycle of Shilnikov type is enough to predict the existence of a countable infinity of suspended spiralling horseshoes. Their existence does not

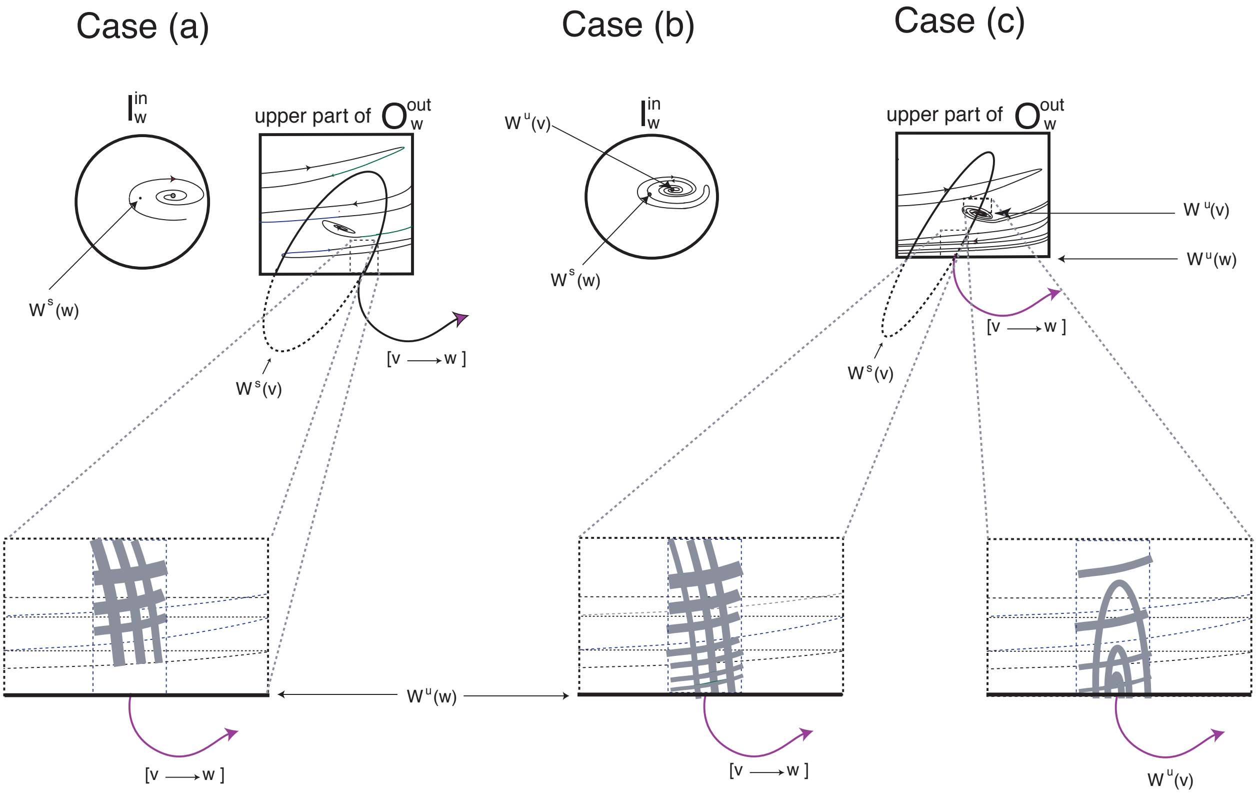

FIGURE 11. There is a sequence of values of λ_2 for which horseshoes with infinitely many *legs* appear and disappear as in a dance, being part of a very complex spiralling network of sets near the ghost of $\Sigma^\star$. (a) A segment in $I^{in}_{\mathbf{v}}$ transverse to $W^s(\mathbf{v})$ is mapped into a spiral in $I^{in}_{\mathbf{w}}$. When λ_2 varies, branches of this curve get closer to $W^s(\mathbf{w})$ and its image makes more turns around $O^{out}_{\mathbf{w}}$, creating horseshoes with more strips. (b) When one of the arms of the double spiral $W^u(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$ touches $W^s(\mathbf{w})$ a homoclinic loop at $\mathbf{w}$ is created. (c) The double spiral $W^u(\mathbf{w}) \cap I^{in}_{\mathbf{w}}$ is mapped into a curve in $O^{out}_{\mathbf{w}}$ that winds a few times around the cylinder and then spirals into a point in $W^u(\mathbf{v})$. A stable homoclinic loop at $\mathbf{v}$ is created when the spiral centre touches the closed curve $W^s(\mathbf{v}) \cap O^{out}_{\mathbf{w}}$. This may happen for the same parameter values that yield the homoclinic at $\mathbf{w}$.

require breaking the homoclinic connection as in the case of saddles whose linearisation has only real eigenvalues. The study of spiralling sets has not attracted as much attention as the Lorenz attractor because, in general, it is difficult to understand the topology associated to non-real eigenvalues.

The spiralling set Λ of Theorem 3 is remarkably different from the robustly transitive sets described by Araújo and Pacífico [6] and Morales *el al* [24]. This is easily seen using assertion (3) of Theorem 7 to obtain, for arbitrarily small λ_2, a homoclinic cycle to $\mathbf{w}$ of Shilnikov type with the expanding eigenvalue greater than the contracting one. This induces the appearance of an arbitrarily large number of attracting or repelling periodic solutions. If Λ were robustly transitive, this would imply that $\mathbf{X}_1$ could be C^1–approximated by vector fields exhibiting either attracting or repelling sets, which would be a contradiction.

Although the spiralling set is not robustly transitive, by Theorem 2 it is persistent under $\mathbf{Z}_2$–symmetric perturbations. The fact that these wild spiral sets contain equilibria makes them

similar to Lorenz-like attractors described by Rovella [29]. Both examples show suspended horseshoes, coexisting with equilibria and periodic solutions. In our example, suspended horseshoes are present for an open set of parameters whereas in Rovella's example they appear for a set of parameters with positive Lebesgue measure. Moreover, in our example, the suspended horseshoes coexist with heteroclinic tangencies and these in turn give rise to attracting periodic orbits that may be situated in the gaps of the horseshoes. The attractor splits into infinitely many components, whereas Rovella's example satisfies Axiom A for a set of parameters with positive Lebesgue measure.

A generic perturbation of the partially symmetric vector field $\mathbf{X}_1$ still has some spiralling structure. It contains a finite sequence of topological horseshoes semiconjugate to full shifts over an alphabet with a finite number of symbols, instead of an infinite sequence. The dynamics near these specific networks can be lifted to larger networks.

The properties of the family $\mathbf{X}$ that we have described depend strongly on the orientation around the heteroclinic connection $[\mathbf{v} \to \mathbf{w}]$. If the rotations inside V and inside W had opposite orientations, for some trajectories these two rotations in V and W would cancel out. Newhouse phenomena would occur near a Bykov cycle of this type, and hence infinitely many attractors. A systematic study of this case is in preparation.

A lot more needs to be done before we understand well the dynamics of systems close to symmetry involving either saddle-foci or rotating nodes. Besides the interest of the study of the dynamics arising in generic unfoldings of an attracting heteroclinic network, its analysis is important because in the fully non-equivariant case the explicit analysis of the first return map seems intractable. In this article we were able to predict qualitative features because the non-symmetric dynamics is close to symmetry.

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Appendix A. Transversality of invariant manifolds

A.1. **Melnikov method revisited.** Melnikov [22] studied a method to find the transverse intersection of the the invariant manifolds for a time periodic perturbation of a homoclinic cycle. The pioneer idea of Melnikov is to make use of the globally computable solutions of the unperturbed system the computation of perturbed solutions. We start this appendix with a short description of this theory applied to saddle-connections. For a detailed proof for the homoclinic case, see Guckenheimer and Holmes [14, section 4.5].

If $X \in \mathbf{R}^2, t \in \mathbf{R}$ and $0 < \varepsilon \ll 1$, consider the planar system:

$$\dot{X} = f(X) + \varepsilon g(X, t) \tag{A.4}$$

such that:

- there exists $T > 0$ such that $g(X, t) = g(X, t + T)$ for all t *ie* g is T-periodic;
- for $\varepsilon = 0$, the flow has a heteroclinic connection Γ_0 associated to two hyperbolic points p_0 and p_1;
- the unstable manifold of p_0 coincides with the stable manifold of p_1.

Associated to the system (A.4), taking $S^1 \cong \mathbf{R}/T$, we define the suspended system:

$$\dot{X} = f(X) + \varepsilon g(X, \theta), \quad \dot{\theta} = 1, \quad (X, \theta) \in \mathbf{R}^2 \times S^1.$$

With the above assumptions, $\{p_0\} \times S^1$ and $\{p_1\} \times S^1$ are hyperbolic periodic solutions for the suspended flow, whose invariant manifolds $\Gamma_0 \times S^1$ coincide. Since the limit cycles are hyperbolic, their hyperbolic continuation is well defined for $\varepsilon \neq 0$; hereafter we denote them

by $\{p_0^\varepsilon\} \times S^1$ and $\{p_1^\varepsilon\} \times S^1$. Under general conditions (without symmetry for example), the heteroclinic connection $\Gamma_0 \times S^1$ is not preserved and for a non-empty open set in the parameter space, the invariant manifolds meet transversely. Their intersection consists of a finite number of trajectories.

The splitting of the stable and unstable manifolds on a transverse cross section Σ for the perturbed system is measured by the *Melnikov function*:

$$M(t_0) = \int_{-\infty}^{+\infty} f(q_0(t)) \wedge g(q_0(t), t + t_0). \exp\left(- \int_0^t trDf(q_0(s))ds \right) dt, \tag{A.5}$$

where $q_0(t)$ is the parametrisation of the solution of the unperturbed system (A.4) starting at $t_0 = 0$ on Γ_0. Recall that the *wedge product* in $\mathbf{R}^2$ of two vectors (u_1, u_2) and (v_1, v_2) is simply given by $u_1 v_2 - u_2 v_1$. When f is hamiltonian, then $trDf(q_0(t)) = 0$ and the Melnikov function is simpler. The main result we will use is the following:

Theorem 8 (Bertozzi [12], Melnikov [22], adapted)**.** *Under the above conditions, and for $\varepsilon > 0$ sufficiently small, if $M(t_0)$ has simple zeros, then $W^u(p_0^\varepsilon)$ and $W^s(p_1^\varepsilon)$ intersect transversely.*

This result is important because it allows to prove the existence of transverse (and hence robust) homo and heteroclinic connections as we proceed to do.

A.2. **Proof of transversality.**

Theorem 9. *If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the two dimensional invariant manifolds of the equilibria* $\mathbf{w}$ *and* $\mathbf{v}$ *of* $\mathbf{X}(X, \lambda_1 \lambda_2)$ *intersect transversely in* $\mathbf{S}^3$ *along one-dimensional orbits.*

Proof. In spherical coordinates

$$x_1 = r \sin\phi \ \sin\theta \cos\varphi \qquad x_2 = r\sin\phi \ \sin\theta \sin\varphi \qquad x_3 = r\cos\phi \sin\theta \qquad x_4 = r\cos\theta$$

equations (2.3) restricted to $\mathbf{S}^3$ can be written as:

$$\begin{aligned} \dot\theta &= \alpha_1 \sin\theta\cos(2\phi) + \tfrac{\alpha_2}{2}\sin(2\theta) + \tfrac{\lambda_1}{2}\sin^2(\phi)\sin^2(\theta)\cos(\phi)\sin(2\varphi) \\ \dot\phi &= -\alpha_1\cos(\theta)\sin(2\phi) - \tfrac{\lambda_1}{4}\sin^3(\phi)\sin(2\theta)\sin(2\varphi) \\ \dot\varphi &= 1 \end{aligned} \tag{A.6}$$

From the equation $\dot\varphi = 1$, we get $\varphi(t) = t$, $t \in \mathbf{R}$ and (A.6) is reduced to a three-dimensional non-autonomous differential equation of the form:

$$\begin{aligned} \dot\theta &= f_1(\theta,\phi) + \lambda_1 g_1(\theta,\phi,t) \\ \dot\phi &= f_2(\theta,\phi) + \lambda_1 g_2(\theta,\phi,t) \end{aligned}$$

where

$$\begin{aligned} f_1(\theta,\phi) &= \alpha_1\sin\theta\cos(2\phi) + \tfrac{\alpha_2}{2}\sin(2\theta) & f_2(\theta,\phi) &= -\alpha_1\cos(\theta)\sin(2\phi) \\ g_1(\theta,\phi,t) &= \tfrac{1}{2}\sin^2(\phi)\sin^2(\theta)\cos(\phi)\sin(2t) & g_2(\theta,\phi,t) &= -\tfrac{1}{4}\sin^3(\phi)\sin(2\theta)\sin(2t)\ . \end{aligned}$$

The maps g_1 and g_2 are periodic in t of period π. For the Melnikov function $M(t_0)$ defined in (A.5) we write $f = (f_1, f_2)$ and $g = (g_1, g_2)$. The parametrisation $q_0(t)$ of the connection $[\mathbf{w} \to \mathbf{v}]$ in the unperturbed system, $\lambda_1 = 0$, is defined by $\phi = \frac{\pi}{2} + k\pi$, $k \in \{0, 1\}$. Thus, in the unperturbed system, the connections $[\mathbf{w} \to \mathbf{v}]$ are parametrised by:

$$q_0^1(t) = \left(\theta(t), \frac{\pi}{2}\right) \text{ and } q_0^2(t) = \left(\theta(t), \frac{3\pi}{2}\right).$$

Therefore, for $k \in \{0, 1\}$, we have:

$$\begin{aligned} f_1(q_0^i(t)) &= (-1)^k\alpha_1\sin(\theta(t)) + \tfrac{\alpha_2}{2}\sin(2\theta(t)) & f_2(q_0^i(t)) &= 0 \\ g_1(q_0^i(t), t+t_0) &= 0 & g_2(q_0^i(t), t+t_0) &= (-1)^{k+1}\tfrac{1}{4}\sin(2\theta(t))\sin(2(t+t_0))\ . \end{aligned}$$

To see that the integral $M(t_0)$ converges, note that the exterior product in the definition of the Melnikov function $M(t_0)$ is bounded since it is given by:

$$\begin{aligned} &f(q_0^i(t)) \wedge g(q_0^i(t), t+t_0) = \\ &= \left[-\alpha_1 \sin(\theta(t)) + (-1)^{k+1}\tfrac{\alpha_2}{2}\sin(2\theta(t))\right]\tfrac{1}{4}\sin(2\theta(t))\sin(2(t+t_0)) \end{aligned}$$

and

$$trDf(q_0(s)) = \alpha_1 \cos(\theta(s)) + \alpha_2 \cos(2\theta(s))$$

hence the Melnikov integral $M(t_0)$ does not depend on λ_1. Since it has been shown in Aguiar *et al* [3, lemma 16] that for any $r > 0$, the integral

$$\int_{-\infty}^{+\infty} \exp\left(-\int_0^t \alpha_2 r\cos(\theta(s)) + \alpha_3 r^2 \cos(2\theta(s))ds\right)dt$$

converges, the convergence of $M(t_0)$ follows. Finally from the proof of Proposition 12 in Aguiar *et al* [3] it follows that the roots of $M(t_0)$ are simple, completing the proof of theorem 9. □

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